# Full waveform inversion and triangular waveform adapted meshes using Firedrake

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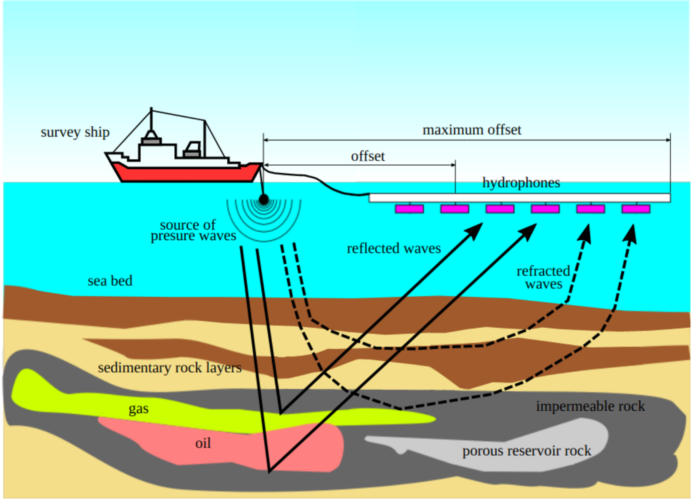
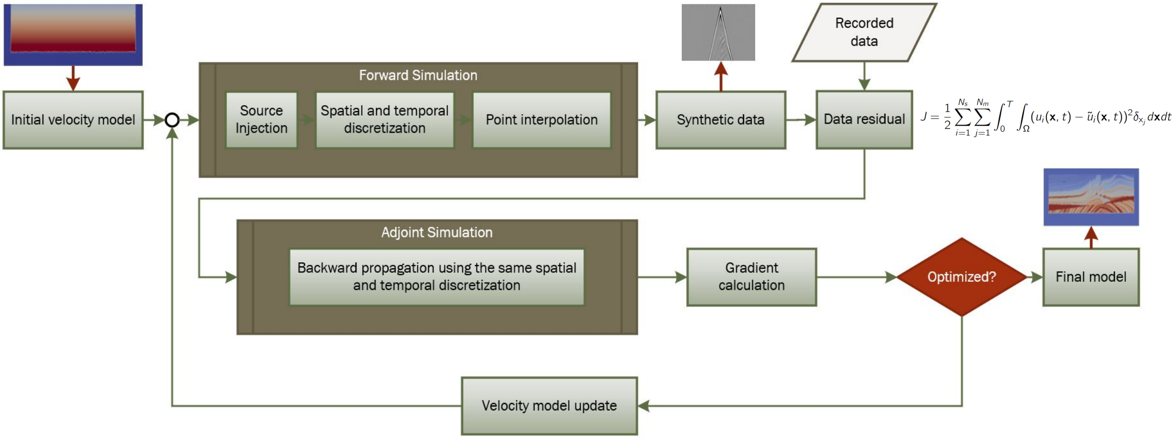
# Full waveform inversion




A typical marine seismic survey

# Full waveform inversion




Iterative scheme used to run a full waveform inversion with Spyro/Firedrake

# Acoustic equation with PML

$$\frac{\partial^2 u}{\partial t^2} - \nabla \cdot (c^2 \nabla u) - \nabla \cdot \mathbf{p} + \Psi_1 \frac{\partial u}{\partial t} + \Psi_3 u + \det \Psi_1 \omega = f, \quad \Rightarrow \text{Wave eq. for pressure } u(\mathbf{x}, t)$$

$$\frac{\partial \mathbf{p}}{\partial t} + \Psi_1 \mathbf{p} + \Psi_2 (c^2 \nabla u) - \Psi_3 (c^2 \nabla \omega) = \mathbf{0}, \quad \Rightarrow \text{PML eqs. for aux. var. } \mathbf{p}(\mathbf{x}, t)$$

$$\frac{\partial \omega}{\partial t} = u, \quad \Rightarrow \text{PML eq. for aux. var. } \omega(\mathbf{x}, t)$$

**Time scheme**: Explicit FD $\Rightarrow$ only Mass-Matrices to invert:

$$\frac{\partial^2 u}{\partial t^2} \Rightarrow M_u, \quad \Psi_1 \frac{\partial u}{\partial t} \Rightarrow M_{u_1}, \quad \frac{\partial \mathbf{p}}{\partial t} \Rightarrow M_p \quad \frac{\partial \omega}{\partial t} \Rightarrow M_\omega$$

see Kaltenbacher et al, JCP (2013) or Grote and Sim, preprint (2010) for PML

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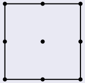
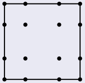
# 2D Elements

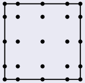
## Spectral quadrilateral finite elements with GLL nodes



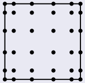
(a) $P = 2$



(b) $P = 3$



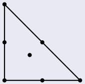
(c) $P = 4$



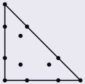
(c) $P = 5$

(Karniadakis and Sherwin, 2013; Patera, 1984)

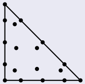
## Mass-lumped simplicial finite elements (KMV)



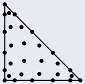
(a) $KMV2tri$



(b) $KMV3tri$



(c) $KMV4tri$

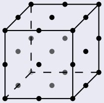


(c) $KMV5tri$

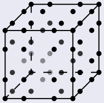
(Kong, Mulder, and Veldhuizen, 1999)

# 3D Elements

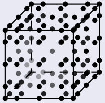
## Spectral hexahedral finite elements with GLL nodes



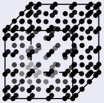
(a) $P = 2$



(b) $P = 3$



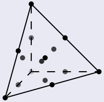
(c) $P = 4$



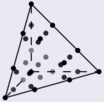
(c) $P = 5$

(Karniadakis and Sherwin, 2013; Patera, 1984)

## Mass-lumped simplicial finite elements (KMV)



(a) *KMV2tet*



(b) *KMV3tet*

(Kong, Mulder, and Veldhuizen, 1999; Geevers, Mulder, and Vegt, 2018)

# Element usage in Firedrake

## We need to get the quadrature rule from FInAT

```
from firedrake import *
import finat
kmv_rule = finat.quadrature.make_quadrature(
        V.finat_element.cell, V.ufl_element().degree(), "KMV"
)
```

## Then, we use it for the mass matrix assembly

```
V = FunctionSpace(mesh, "KMV", 3)
u = TrialFunction(V)
v = TestFunction(V)

m1 = ((u - 2.0 * u_n + u_nm1) / Constant(dt ** 2)) * v * dx(kmv_rule)
```

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# spyro: Acoustic wave modeling

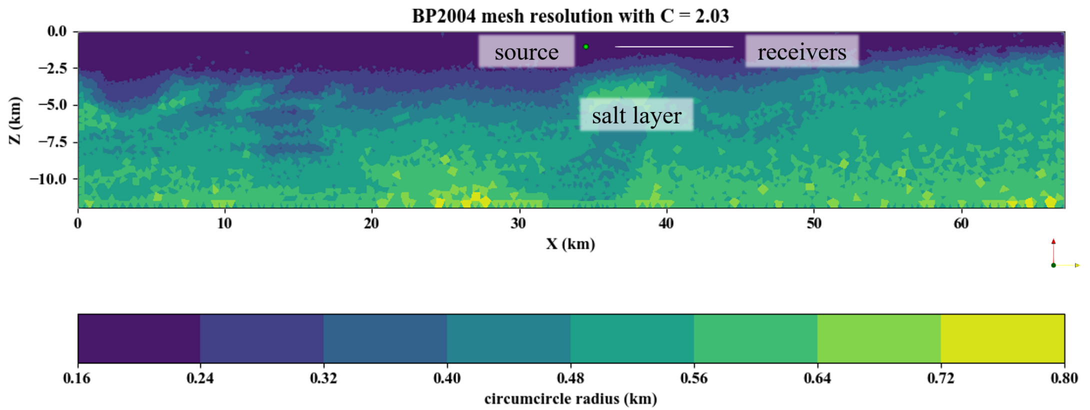
## Features

- Based on Firedrake (Rathberger et al. 2017)
  - Mass-lumped triangular and tetrahedral elements (KMV)
  - Spectral quadrilateral and hexahedral elements
- Perfectly Matched Layer (PML) to absorb reflected waves
- Capable of doing the complete FWI loop
  - Mesh-independent functional gradient using the optimize-then-discretize approach
  - Sparse interpolation and injection with point and force sources
- Automatic mesh generation with SeismicMesh (Roberts, et al. 2021a)

## Link

https://github.com/krober10nd/Spyro

# spyro works with heterogeneous wave velocity




Example of mesh generated with SeismicMesh

# Mesh design: cells per wavelength

Experiment setup based on:

- Ricker source with peak frequency of 5 Hz;
- Homogeneous velocity model of constant $c = 1,429$ km/s;
- Square grid of 36 point receivers that don't coincide with mesh nodes;
- Results compared with a reference solution in a fine high- order mesh;
- Based on experiments by Lyu et at. (2020).

In order to calculate:

- $C$: cells per wavelength necessary to inhibit numerical error
- $G$: degrees-of-freedom per wavelength necessary to inhibit numerical error.

$$E = \sqrt{\frac{\sum_{r=1}^{N_r} \int_0^t {}_f (p_r - p_{r_ref})^2 dt}{\sum_{r=1}^{N_r} \int_0^t {}_f p_{r_ref}^2 dt}} \quad (1)$$

# Mesh design: cells per wavelength

| Element | Homogeneous minimum $G$ | Homogeneous minimum $C$ |
|---|---|---|
| $KMV1tri$ | DNF | DNF |
| $KMV2tri$ | 10.1 | 5.85 |
| $KMV3tri$ | 7.86 | 3.08 |
| $KMV4tri$ | 7.36 | 2.22 |
| $KMV5tri$ | 7.88 | 1.69 |

Table: Results from Roberts et al. (2021b), detailing the necessary cells per wavelength to use in waveform adapted meshes. For heterogeneous velocity models we propose an increase of 20%.

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# Strong scaling on the forward problem

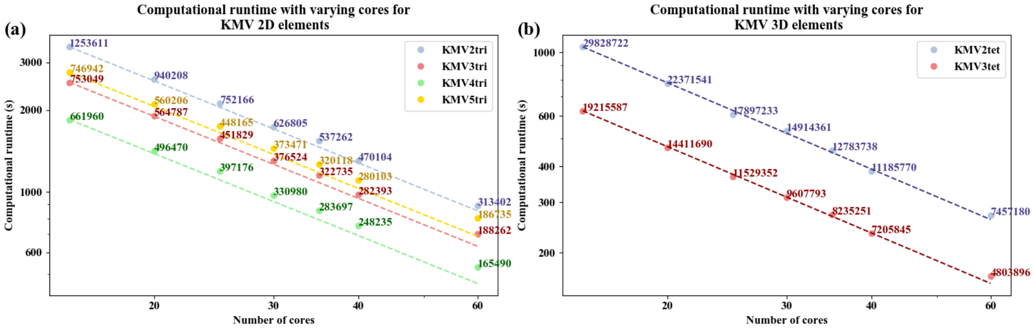
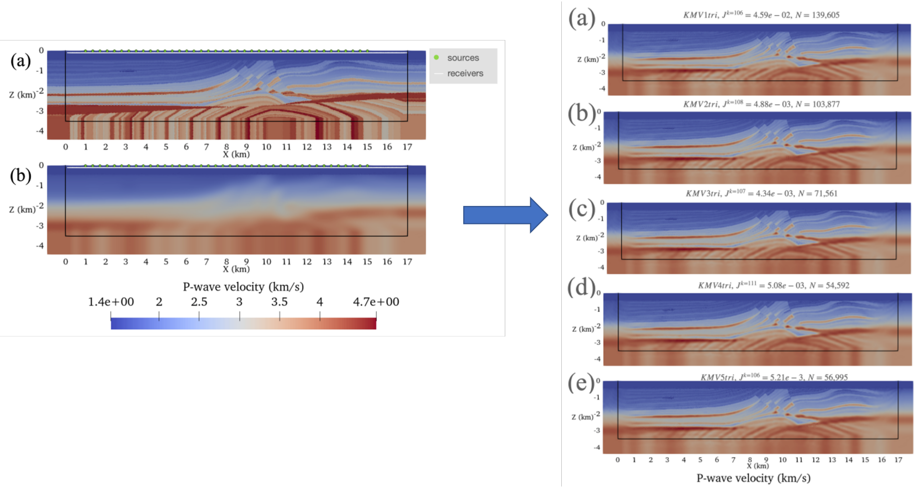



Figure: The performance of the forward.py wave propagator was assessed in the following benchmark 2D (a) and 3D meshes (b), where the ideal strong scaling line for each KMV element is represented as dashed and the number of degrees of freedom per core is annotated. For the 2D benchmark, the domain spans a physical space of 110 km by 85 km. A domain of 8 km by 8 km by 8 km was used in the 3D case. Both had a 0.287 km wide PML included on all sides of the domain except the free surface and a uniform velocity of 1.43 km/s.

# 2D Experiments




From Roberts et al. (2021b).

# Comparison of memory storage 2D

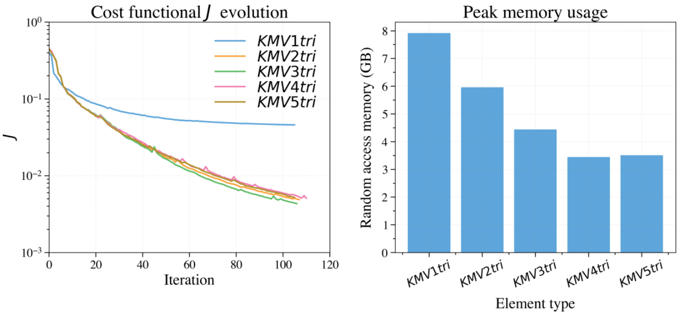



Figure: From Roberts et al. (2021b).

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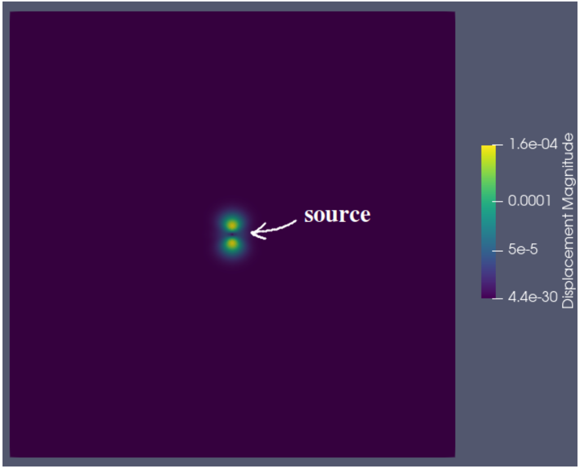
# Elastic FWI - forward model so far...

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \mathrm{div}\left(\lambda\, \mathrm{tr}\left(\boldsymbol{\epsilon}\right) \mathbf{I} + 2\mu\, \boldsymbol{\epsilon}\right) + \mathbf{f},$$
$$\boldsymbol{\epsilon} = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^{\intercal}\right). \tag{2}$$

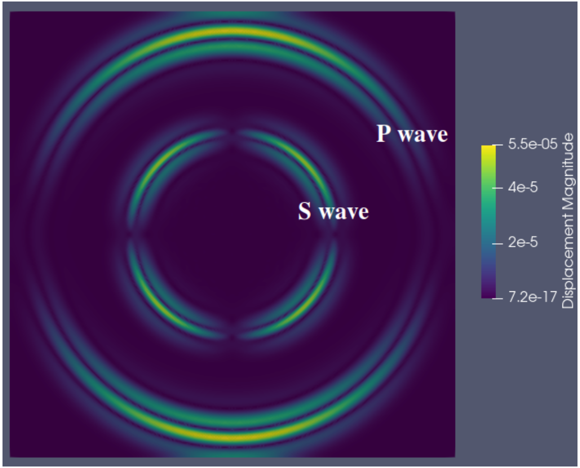
- $\mathbf{u}$: displacements
- $\rho$: density
- $\lambda$: 1st Lamé parameter (to be inverted by FWI)
- $\mu$: 2nd Lamé parameter (to be inverted by FWI)

$$\text{wave speeds} \begin{cases} c_p = \sqrt{\dfrac{\lambda + 2\mu}{\rho}} & \text{(P wave)} \\ \\ c_s = \sqrt{\dfrac{\mu}{\rho}} & \text{(S wave)} \end{cases} \tag{3}$$

# Elastic FWI - forward model, simple test

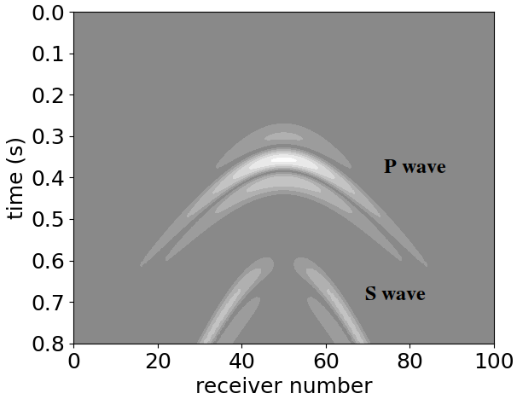


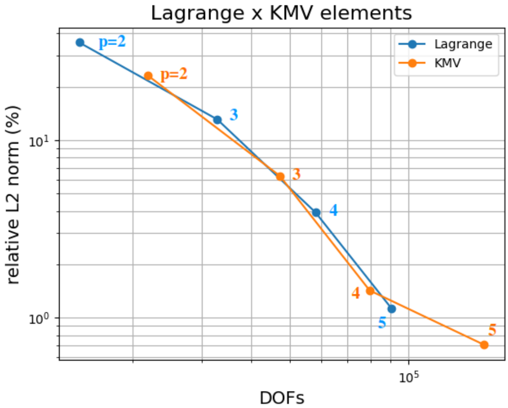

front wave, t=0.1 s




front wave, t=0.8 s

# Elastic FWI - forward model, simple test



# Ongoing work

- Weight-Adjusted Discontinuous Galerkin
- Multiple time integration schemes using irksome

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*arXiv preprint arXiv:2108.10946*.

Thank you for listening!

## Acknowledgements

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# spyro: Acoustic wave modeling

## Features

- Based on the Firedrake automated system for solution of PDEs using FEM.
  - Mass-lumped 2D and 3D using triangular and tetrahedral elements.
  - Spectral quadrilateral and hexahedral elements.
- Perfectly Matched Layer (PML) to absorb reflected waves in both 2D and 3D.
- Capable of doing the complete FWI loop:
  - Mesh-independent functional gradient using the optimize-then-discretize approach.
  - Sparse interpolation and injection with point sources or force sources.
- Integrated with automatic mesh generation software SeismicMesh (Roberts, et al. 2020) for triangles and tetrahedrals.

## Link

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