A matrix-free high-order spectral implementation of the acoustic wave equation with a perfectly matched layer

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Full waveform inversion

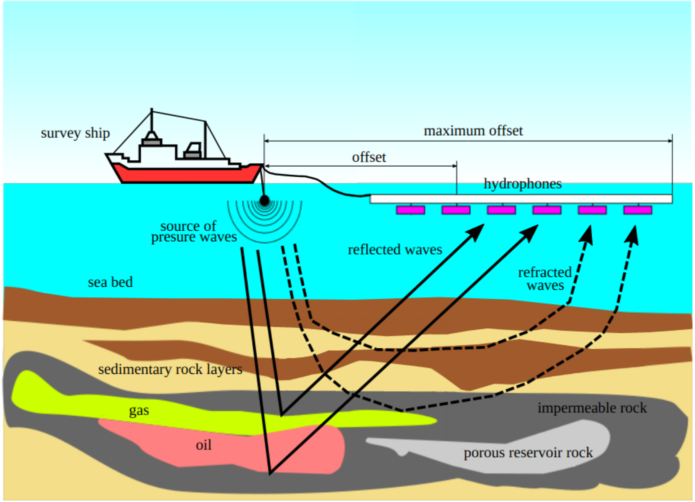

Figure: A marine seismic survey.

Full waveform inversion

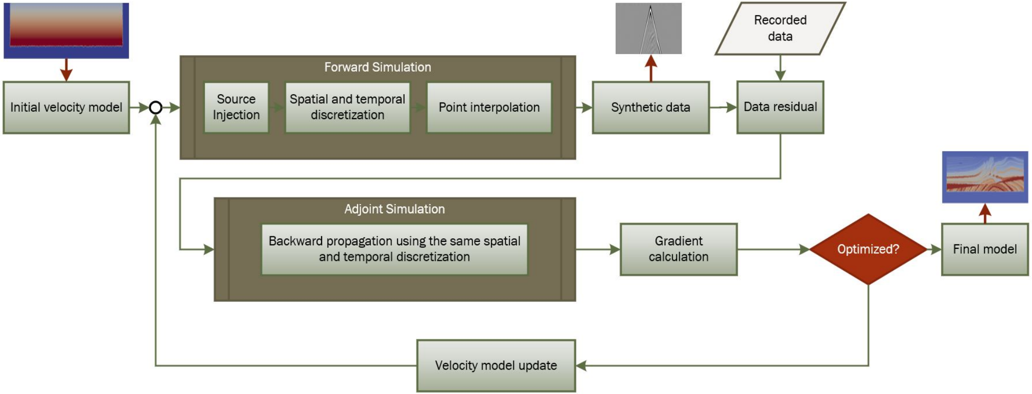

Figure: Iterative scheme used to do full waveform inversion. Images generated with open-source software spyro by authors.

Acoustic equation with PML

$$\frac{\partial^2 u}{\partial t^2} - \nabla \cdot (c^2 \nabla u) - \nabla \cdot \mathrm{p} + \Psi_1 \frac{\partial u}{\partial t} + \Psi_3 u + \det \Psi_1 \omega = f, \quad \Rightarrow \text{Wave eq. for pressure } u(\mathrm{x}, t)$$

$$\frac{\partial \mathrm{p}}{\partial t} + \Psi_1 \mathrm{p} + \Psi_2 (c^2 \nabla u) - \Psi_3 (c^2 \nabla \omega) = 0, \quad \Rightarrow \text{PML eqs. for aux. var. } \mathrm{p}(\mathrm{x}, t)$$

$$\frac{\partial \omega}{\partial t} = u, \quad \Rightarrow \text{PML eq. for aux. var. } \omega(\mathrm{x}, t)$$

Time scheme: Explicit FD $\Rightarrow$ only Mass-Matrices to invert:

$$\frac{\partial^2 u}{\partial t^2} \Rightarrow M_u, \quad \Psi_1 \frac{\partial u}{\partial t} \Rightarrow M_{u_1}, \quad \frac{\partial \mathrm{p}}{\partial t} \Rightarrow M_p \quad \frac{\partial \omega}{\partial t} \Rightarrow M_\omega$$

W.L.O.G. we will present the method on M_u

see Kaltenbacher et al, JCP (2013) or Grote and Sim, preprint (2010) for PML

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Matrix-free methods

Benefits

- Reduced memory usage (Kirby and Mitchell, 2018)
- Comparable algorithmic complexity for simplex elements without sum-factorization (Kirby and Mitchell, 2018)
- Improved complexity for sum-factorized tensorial elements (Kirby and Mitchell, 2018)

Matrix-free methods

For our mass matrix

$$(M_u \cdot u)_i = \sum_{j=1}^{N_{dof}} M_{u_ij} u_j = \sum_{j=1}^{N_{dof}} \int_\Omega (\phi_i \phi_j dx) u_j =$$

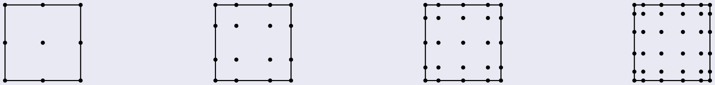
$$\sum_{j=1}^{N_{dof}} a(\phi_i, \phi_j) u_j = a(\phi_i, \sum_{j=1}^{N_{dof}} \phi_j u_j) = a(\phi_i, u)$$

When to use

When the algorithm for finding the solution of a linear system only relies on the action of the operator, therefore not requiring matrix assembly (Homolya, Kirby and Ham 2017).

2D Elements

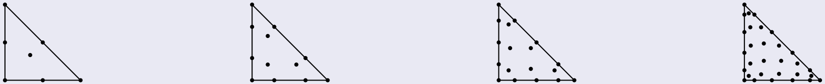
Spectral quadrilateral finite elements with GLL nodes.



(a) $P = 2$ (b) $P = 3$ (c) $P = 4$ (c) $P = 5$

Figure: Some two-dimensional spectral elements with GLL nodes (Karniadakis and Sherwin, 2013).

Mass-lumped (KMV) simplicial finite elements.



(a) $KMV2tri$ (b) $KMV3tri$ (c) $KMV4tri$ (c) $KMV5tri$

Figure: Elements from Kong, Mulder and Veldhuizen (1999) and Geevers, Mulder and Vegt (2018)

KMV elements

Comparison with GRL with collapsed node SEM

With KMV elements we have an increase in the number of nodes per element, however we now have a diagonal mass matrix.

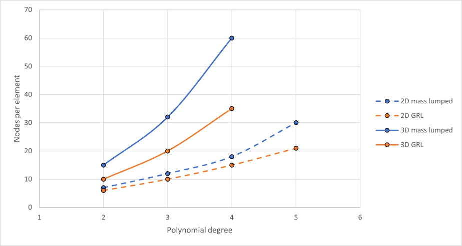
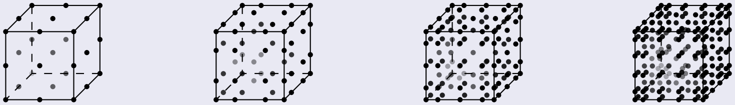


Figure: Comparing nodes per element when using the quadrature for mass-lumping from Geevers, Mulder and Vegt (2019) and the quadrature scheme of Gaus-Radau-Legendre in 2D and 3D simplices (Karniadakis and Sherwin, 2013).

3D Elements

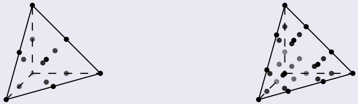
Spectral hexahedral finite elements with GLL nodes.



(a) $P = 2$ (b) $P = 3$ (c) $P = 4$ (c) $P = 5$

Figure: 3D spectral elements with GLL nodes.

KMV mass-lumped simplicial finite elements.



(a) $KMV2tet$ (b) $KMV3tet$

Figure: 3D mass-lumped elements

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spyro: Acoustic wave modeling

Features

- Based on the Firedrake for solution of PDEs using FEM (Rathberger et al. 2017).
  - Mass-lumped 2D and 3D using triangular and tetrahedral elements.
  - Spectral quadrilateral and hexahedral elements.
- Perfectly Matched Layer (PML) to absorb reflected waves in both 2D and 3D.
- Capable of doing the complete FWI loop:
  - Mesh-independent functional gradient using the optimize-then-discretize approach.
  - Sparse interpolation and injection with point sources or force sources.
- Integrated with automatic mesh generation software SeismicMesh (Roberts, et al. 2020) for triangles and tetrahedrals.

Link

https://github.com/krober10nd/Spyro

spyro works with heterogeneous velocity and unstructured 2D mesh

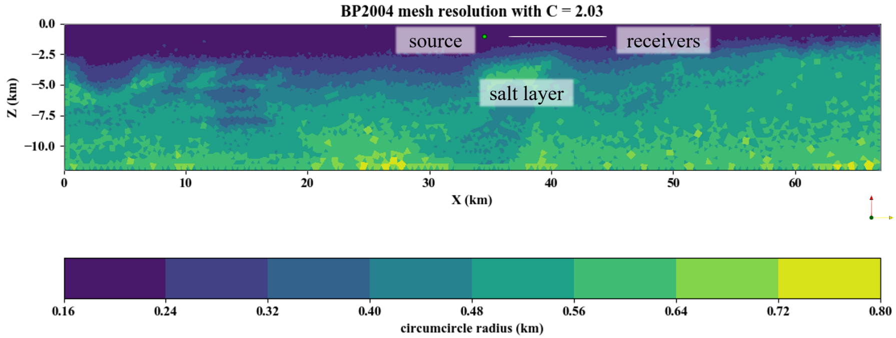

Figure: Mesh generated with automatic mesh software SeismicMesh and with source injection and point interpolation of the solution to the receivers in the same space of the finite elements used to discretize the domain. The velocity model was taken from BP2004 (Billette and Brandsberg-Dahl, 2004).

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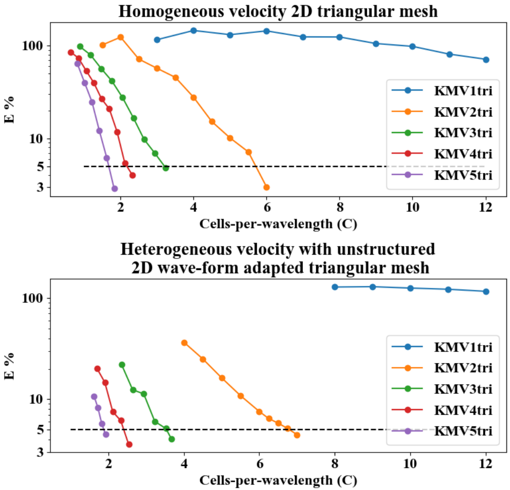
2D Experiments

In order to compare spectral quadrilaterals with mass-lumped triangles

- A Ricker point source was added and an homogeneous velocity experiment was setup based on Lyu et al. (2020).
- Varying C, h-convergence was analyzed for different polynomial degrees based on a reference solution.
- Using a 20% C increase, results were validated with different configurations of heterogeneous velocity model (BP2004 and Marmousi) and more realistic point source and receiver placement.



Comparison of memory storage 2D

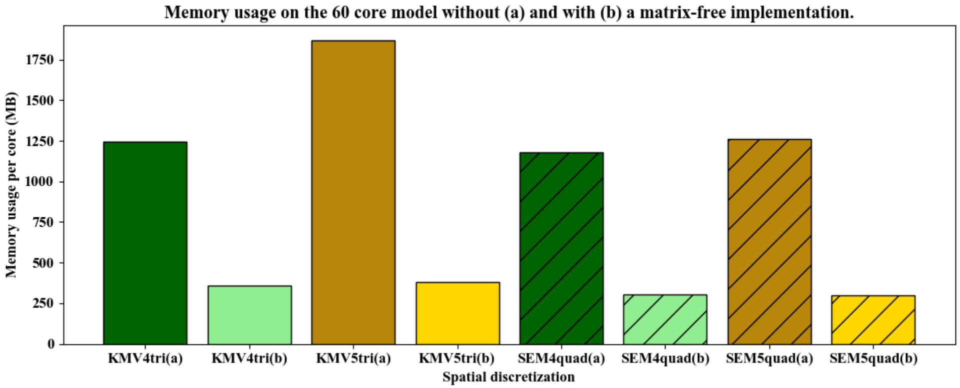


Figure: Comparison of memory storage per core in 2D, in a 60 core simulation, for selected elements.

Strong scaling results 2D

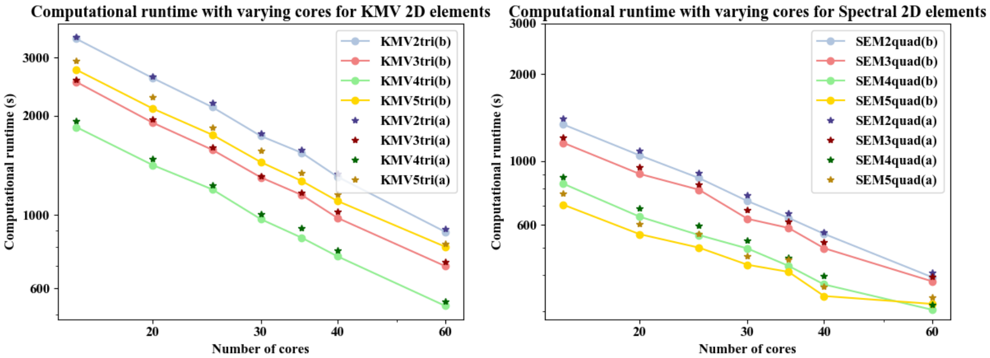


Figure: Comparison of strong scaling while varying core count.

Strong scaling results 2D - heterogeneous

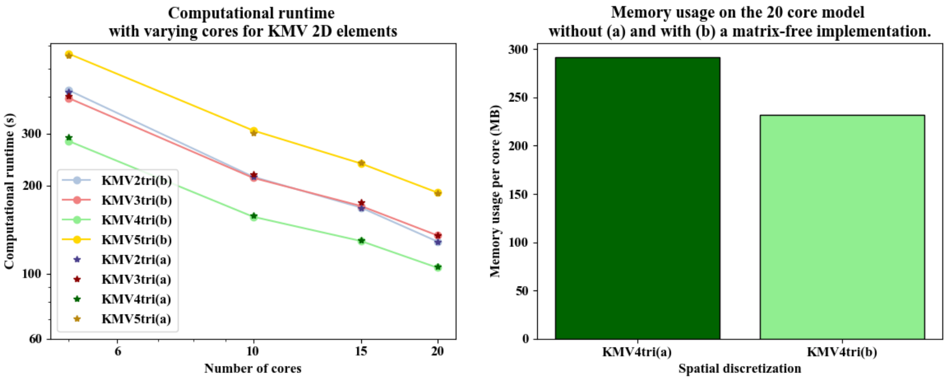


Figure: Comparison of strong scaling while varying core count with cell sizes adapted to material properties of the domain on an unstructured mesh with heterogeneous velocity profile.

Comparison of memory storage and scaling in 3D

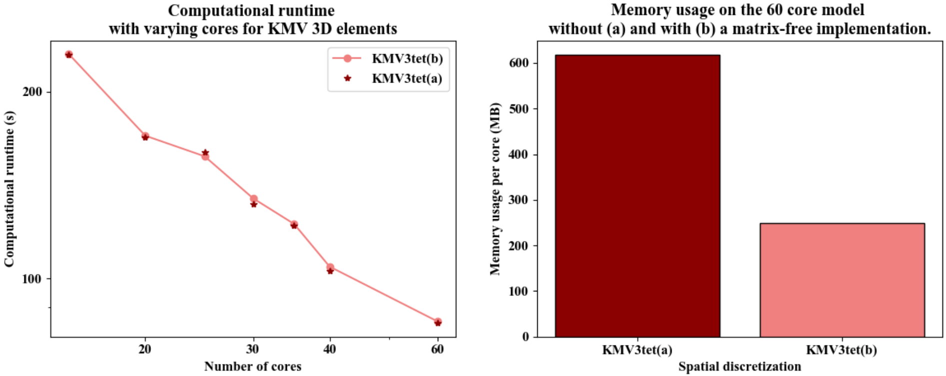


Figure: Comparison of strong scaling while varying core count and memory storage with simulation on 60 cores.

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